Int. J. Solids Structures Vol. 19, No. 7, pp. 653–658, 1983
Printed in Great Britain.

0020-7683/83 \$3.00+.00

THE EQUILIBRIUM STATES OF A HEAVY ROTATING COLUMN

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(*Received* 5 *August* 1982; *in revised form* 13 *December* 1982)

Abstract—A heavy, rotating vertical column is clamped at one end and free at the other end. The stability boundaries are found by both analytical approximations and numerical integration. The problem depends on two non-dimensional parameters: β representing the importance of gravity to rigidity and α representing the importance of rotation to rigidity. Buckled shapes for the different modes are also obtained.

1. INTRODUCTION

Consider a thin, vertical inextensible column of circular cross section and uniform density. Let one end be fixed vertically and the other end free. If the column is hanging from a foundation, the only equilibrium shape is the trivial (vertical) one. If the column is standing on a foundation, depending on the non-dimensional parameter $\beta = \rho g L^3/EI$, other equilibrium shapes exist. Here ρ is the mass per unit length, g is the gravitational acceleration, L is the length of the column, and EI is the flexural rigidity. The (static) stability of a standing column was first investigated by Euler[1] whose result was later corrected by Greenhill[2]. Greenhill concluded that a vertical column cannot buckle if $\beta < 7.95$.

The situation is different when the column begins to rotate about its axis. Centrifugal force now acts as an important destabilizing factor. As we shall see later, even the hanging column becomes unstable if a certain critical rotation speed is attained.

The instability due to the whirling of a rotating shaft was briefly discussed by Love[3]. The large deflections of a rotating shaft with one end free were investigated by Wang[4]. Both sources, however, did not consider gravity effects of the shaft itself. The present paper studies the stability of a heavy rotating column. Particular attention will be paid to the possible nontrivial equilibrium states (bifurcation solutions).

2. FORMULATION

Figure 1 shows the origin of coordinate axes (r', z') situated at the fixed end of a heavy column rotating with constant angular velocity Ω. If rotation and gravity are absent, the column will remain straight on the z' axis. Let s' be the arc length from the origin and θ be the local inclination. A local moment balance on an elemental segment ds' gives

$$m = m + dm + \rho g(L - s') \sin\theta \, ds' + F \cos\theta \, ds'. \tag{1}$$

Here m is the local moment proportional to the local curvature

$$m = EI \frac{d\theta}{ds'} \tag{2}$$

and F is the centrifugal force acting from s' to L

$$F \equiv \int_{s'}^{L} \rho r' \Omega^2 \, ds'. \tag{3}$$

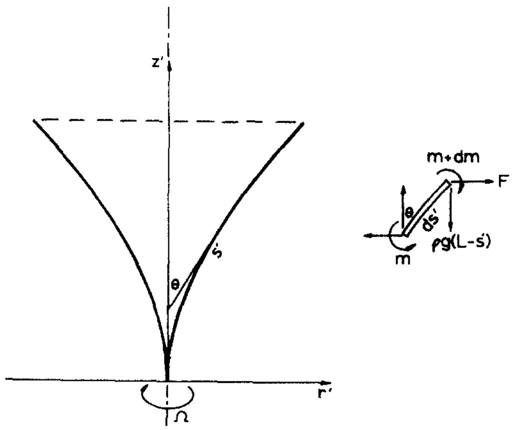

Fig. 1. The rotating heavy column and elemental segment

We shall normalize all lengths by L and drop primes. Equations (1)–(3) yield

$$\frac{d^2\theta}{ds^2} = -\beta(1-s)\sin\theta + \alpha u\cos\theta \tag{4}$$

$$\frac{du}{ds} = r \tag{5}$$

$$u \equiv \int_1^s r\, ds$$

where $\alpha \equiv \rho L^4\Omega^2/EI$ is a non-dimensional parameter representing the relative importance of rotation to flexural rigidity. Geometrical considerations dictate.

$$\frac{dz}{ds} = \cos\theta, \quad \frac{dr}{ds} = \sin\theta. \tag{6}$$

The boundary conditions are

$$\theta(0) = r(0) = z(0) = 0 \tag{7}$$

$$\frac{d\theta}{ds}(1) = 0, \quad u(1) = 0. \tag{8}$$

In the case when the column is hanging down, we can regard β as negative.

3. STABILITY ANALYSIS

The static stability or bifurcation points may be obtained from a linearization of eqns (4)–(8). For infinitesimal θ we can deduce the stability equation

$$\frac{d^4u}{ds^4} = -\beta(1-s)\frac{d^2u}{ds^2} + \alpha u \tag{9}$$

$$\frac{du}{ds}(0) = \frac{d^2u}{ds^2}(0) = u(1) = \frac{d^3u}{ds^3}(1) = 0. \tag{10}$$

Even this linearized eigenvalue problem is extremely difficult to solve. Let us first consider some extreme cases.

If $\beta = 0$ gravity is negligible. Equation (9) yields the general solution

$$u = C_1\sin Js + C_2\cos Js + C_3\sinh Js + C_4\cosh Js \tag{11}$$

where $J \equiv \alpha^{1/4}$ and the C's are constants. Applying the boundary conditions eqn (10) the criterion for non-trivial solution is

$$1 + \cos J \cosh J = 0 \tag{12}$$

The roots are at $J = 1.87510, 4.69409, 7.85476, 10.99554$, etc.

On the other hand, for a non-rotating standing column $\alpha = 0$, $\beta > 0$. Equation (9) and (10) may be written as

$$\frac{d^2\theta}{ds^2} + K^3(1-s)\theta = 0 \tag{13}$$

$$\theta(0) = \frac{d\theta}{ds}(1) = 0 \tag{14}$$

where $K \equiv \beta^{1/3}$. Equation (13) is known as the Stokes equation. The general solution is

$$\theta = C_5\sqrt{(1-s)}J_{-1/3}\left(\frac{2}{3}K^{3/2}(1-s)^{3/2}\right) + C_6\sqrt{(1-s)}J_{1/3}\left(\frac{2}{3}K^{3/2}(1-s)^{3/2}\right). \tag{15}$$

Here $J_{\pm 1/3}$ are Bessel functions of $\pm 1/3$ order. Seeking non-trivial solutions eqns (14) and (15) give

$$J_{-1/3}\left(\frac{2}{3}K^{3/2}\right) = 0. \tag{16}$$

The zeroes are at $K = 1.98635, 3.82557, 5.29566, 6.58432$, etc. The first value gives $\beta = 7.8373$ which is much more accurate than the infinite series value obtained by Greenhill[2]. The hanging column ($\beta < 0$) is always stable when $\alpha = 0$.

We can also consider the limiting case when $\beta \to -\infty$, $\alpha \to \infty$. Physically this represents the rotation of a hanging chain since the stiffness EI approaches zero. Equation (9) reduces to

$$(1-s)\frac{d^2u}{ds^2} + Nu = 0 \qquad N \equiv \Omega^2 L/g. \tag{17}$$

The reduced boundary conditions are

$$u(1) = \frac{du}{ds}(0) = 0. \tag{18}$$

The general solution to eqn (17) is

$$u = C_7\sqrt{(1-s)}J_1(2N^{1/2}(1-s)^{1/2}) + C_8\sqrt{(1-s)}Y_1(2N^{1/2}(1-s)^{1/2}). \tag{19}$$

Here J_1 and Y_1 are Bessel functions of the first and second kind, order 1. Using the boundary conditions we find the condition for non-trivial solution is

$$J_0(2\sqrt{N}) = 0. \tag{20}$$

Thus the eigenvalues are $N = 1.4458, 7.6178, 18.7218, 34.7601$, etc. The relation between K and J is then

$$K = -N^{-1/3}J^{4/3}. \tag{21}$$

4. NUMERICAL RESULTS OF THE STABILITY BOUNDARY

For non-extreme cases, eqns (9) and (10) have to be numerically integrated. We apply a zero-finding method to the eigenvalue determinant of the independent solutions. The solutions

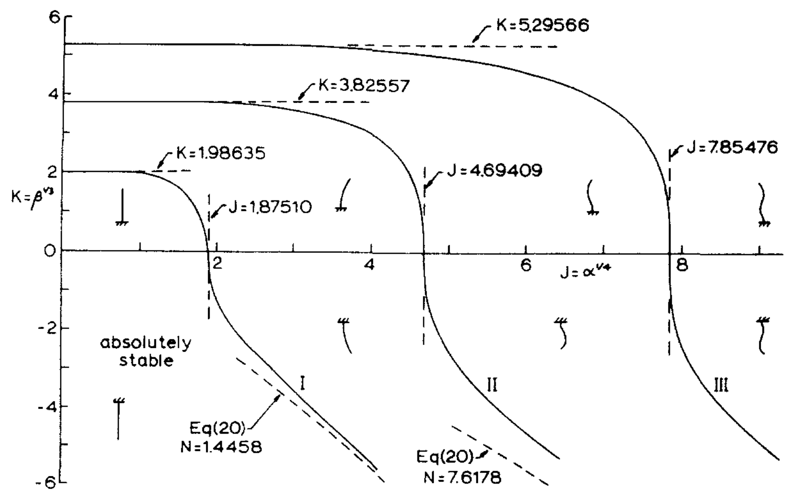

Fig. 2. The stability boundaries. Dashed lines are approximations.

to the linear two-point boundary value problems are obtained by Godunov's reothonormalization algorithm, as implemented and considerably improved in the subroutine SUPORT from Sandia Laboratories[10, 11]. The orthonormalization is done by a *QR* factorization rather than the numerically inaccurate Gram–Schmidt method. The number and placement of the orthonormalization points are chosen dynamically, and the forward integration is a variable step, variable order, Adams PECE algorithm. The driver for SUPORT (also from Sandia Laboratories) is quite easy to use and is fast and accurate. The zero-finding technique applied to the eigenvalue determinant is a sophisticated, robust hybrid method based on bisection and the secant method (subroutine ZEROIN in [5]).

Figure 2 shows the static stability boundaries for the first three modes. To the left of curve *I* only the trivial solution exists, i.e. the column remains vertical or stable in the structural sense. Between curve *I* and curve *II* the first bifurcation has occurred. The bifurcation is a pitchfork type with the trivial solution in unstable equilibrium while the (buckled) bent shape is in stable equilibrium. To the right of curve *II* the second bifurcation has occurred and we may have higher buckling modes on the second bifurcation branch. These curves are important for the design of rotating columns.

Also on Fig. 2, represented by dashed lines, are the various approximations to the stability equations. We see that the approximations are quite satisfactory in their respective regions of validity. The most important stability boundary, curve *I*, is particularly well represented.

5. SOME BUCKLED PROFILES

Once the bifurcation points (points on the stability curves) are determined, the non-trivial solutions on the bifurcation branches can be found from the original nonlinear equations (4)–(8) as follows. For given α and β let

$$\mathbf{y} = \left(\frac{d\theta}{ds}(0), u(0)\right) \tag{22}$$

and $\theta(s; \mathbf{y})$, $r(s; \mathbf{y})$, $u(s; \mathbf{y})$, $z(s; \mathbf{y})$ be the solutions to the initial value problem eqns (4)–(7), (22). Then the original two-point boundary value problem is equivalent to finding a vector $\mathbf{y}$ such that

$$\mathbf{F}(\mathbf{y}) = \left(\frac{d\theta}{ds}(1; \mathbf{y}), u(1; \mathbf{y})\right) = 0. \tag{23}$$

Equation (23) is solved by a combination of homotopy and quasi-Newton methods similar to that described in [6, 7]. For more details on the homotopy method, see [8, 9]. The quasi-Newton computer code used is subroutine HYBRJ1 developed at Argonne National Laboratory (MINPACK-1). In general, the homotopy method is used to obtain a solution, then using this (good) starting point the rest of the branch is computed by the quasi-Newton method. The shapes of the column after bifurcation are then obtained by integrating eqns (4)–(6).

As an illustration, we have computed some representative buckled shapes for $\beta = 4$ ($K = 1.5874$) and various α. The configurations for other values of α and β are similar. Figure 3 shows the equilibrium shapes of the column on the first bifurcation branch (the first mode). The curvature is of one sign. Figure 4 shows those of the second bifurcation branch where the curvature changes sign once. Figure 5 shows the third buckled mode with two sign changes of

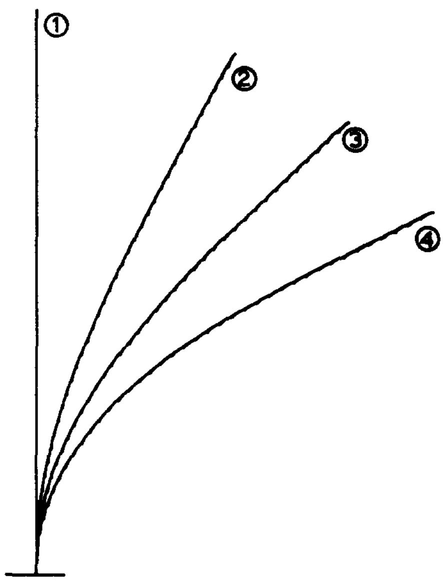

Fig. 3.

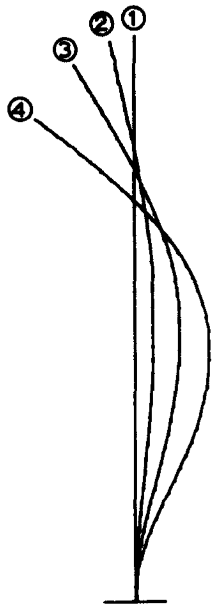

Fig. 4.

Fig. 3. Buckled shapes for the first mode, $\beta = 4$. ① $\alpha \leq 6.06621$, ② $\alpha = 7$, ③ $\alpha = 9$, ④ $\alpha = 13$.

Fig. 4. Buckled shapes for the second mode, $\beta = 4$. ① $\alpha \leq 450.8967$, ② $\alpha = 458$, ③ $\alpha = 494$, ④ $\alpha = 598$.

Fig. 5. Buckled shapes for the third mode, $\beta = 4$. ① $\alpha \leq 3706.619$, ② $\alpha = 3759$, ③ $\alpha = 3988$.

curvature. In practice the column will either not buckle at all (to the left of curve I in Fig. 2) or buckle with the first mode since it has the least potential elastic energy. The higher modes will occur under lateral restrained conditions.

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